

# CAYLEY-TYPE GRAPHS FOR GROUP-SUBGROUP PAIRS

CID REYES-BUSTOS

**ABSTRACT.** In this paper we introduce a Cayley-type graph for group-subgroup pairs and present some elementary properties of such graphs, including connectedness, their degree and partition structure, and vertex-transitivity. We relate these properties to those of the underlying group-subgroup pair. From the properties of the group, subgroup and generating set some of the eigenvalues can be determined, including the largest eigenvalue of the graph. In particular, when this construction results in a bipartite regular graph we show a sufficient condition on the size of the generating sets that results on Ramanujan graphs for a fixed group-subgroup pair. Examples of Ramanujan pair-graphs that do not satisfy this condition are also provided, to show that the condition is not necessary.

## 1. INTRODUCTION

A Ramanujan graph is a  $k$ -regular finite graph with nontrivial eigenvalues  $\mu$  satisfying

$$(1) \quad |\mu| \leq 2\sqrt{k-1},$$

this condition is equivalent to the “Graph Theoretical Riemann Hypothesis” for the Ihara zeta function associated to the graph, see [12] for more information. The above equivalence was first formulated by Sunada in [11]. Infinite families of  $k$ -regular Ramanujan graphs of increasing size are examples of the more general families of expander graphs. In fact, the families of Ramanujan graphs are the expander families that are optimal from the spectral point of view, according to the bounds by Alon and Boppana. Furthermore, the known constructions of families of Ramanujan graphs are essentially number theoretical; for instance, the proof of the Ramanujan property (1) for the original families is related to the Ramanujan conjecture for modular forms of weight 2. The families of expanders have applications to engineering and computer science, among others areas [6].

The original construction of families of Ramanujan graphs was presented by Lubotzky, Phillips and Sarnak in [9], and independently by Margulis in [10]. The former construction consists of Cayley graphs on projective linear groups over a finite field. The use of Cayley graphs allows to utilize the properties of the underlying group and the generating set to determine structural and spectral properties of the graph. For example, the irreducible representations of the underlying group are directly related to the eigenvalues of the graph.

Recently, there has been work on generalizing the notion of group determinant as a function of a group-subgroup pair using the representation theory of

---

*Date:* November 25, 2014.

*2010 Mathematics Subject Classification.* Primary 05C25; Secondary 05C50, 20C99.

*Key words and phrases.* Cayley graph, Group and Subgroup, Ramanujan graphs, Regular graph spectrum, Adjacency matrix.

$\alpha$ -determinants [7]. The resulting wreath determinant for group-subgroup pairs shares some of the properties of the group determinant, but there is a nontrivial relation with the chosen ordering of the group and subgroup elements. Some results on factorization for this determinant can be found on [5], it turns out to be closely related to the representation of the symmetric groups associated with the given group and subgroup and their wreath product.

It was suggested to the author by Masato Wakayama that one could employ the same strategy to extend existing constructions, or to conceive new constructions, of Ramanujan graphs and expanders based on groups, in particular, by generalizing Cayley graphs. In this paper, we apply these ideas to introduce an extension of Cayley graphs for a group-subgroup pair. The motivation is twofold, one is to have further tools to solve problems, in particular the construction of Ramanujan graphs; and to continue the work on this incipient group-subgroup study philosophy.

The main purpose of this paper is to introduce the concept of group-subgroup pair graph based on these considerations and show its elementary properties. In particular, the degree of the vertices of the pair graph is determined by the coset structure of the subgroup and the chosen generating set. The organization of the paper is as follows. In Section 2, the definition, examples and basic properties are presented. Namely, the determination of the degrees of the vertices of the graph, and the conditions for the graph to be regular. In Section 3, conditions for connectedness of the graph are completely determined, along with the number and properties of connected components for disconnected graphs. Additionally, sufficient conditions for the graphs to be bipartite and the notion of trivial eigenvalues are introduced for the group-subgroup pair graphs. In Section 4 we limit the discussion to the regular graph case and assert a symmetry relation between the spectrum for different choices of generating set for fixed group and subgroup, this result is applied to give a sufficient condition on the size of the generating set that guarantees that the resulting graphs are Ramanujan graphs. We provide two examples of Ramanujan graphs obtained by using this condition. The common theme throughout this paper is to compare and relate the results on group-subgroup pair graphs with results on Cayley graphs.

## 2. DEFINITION AND BASIC PROPERTIES

In this section we introduce the group-subgroup pair graph, a Cayley-type graph construction for group-subgroup pairs; relate the definition with the group matrix and establish some elementary properties of the graphs.

First, we introduce the conventions and notation used in this paper. All groups are assumed to be finite unless otherwise stated and  $e$  always represents the identity of a given group  $G$ . For a subgroup  $H$  of  $G$ , all cosets are assumed to be right cosets and we say that  $a$  is incongruent to  $b$  modulo  $H$  when  $Ha \neq Hb$ . A subset  $X \subset G$  is said to be symmetric if  $X^{-1} = X$ . For a given group  $G$  and symmetric subset  $S$  we denote the corresponding Cayley graph by  $\mathcal{G}(G, S)$ . The characteristic function of a subset  $X \subset G$  is denoted  $\delta_X$  and the notation  $[k]$  for  $k \in \mathbb{N}$  is used for the set  $\{1, 2, \dots, k\}$ .

### 2.1. Definition and examples.

**Definition 2.1.** Let  $G$  be a group,  $H$  a subgroup and  $S \subset G$  a subset such that  $S \cap H$  is a symmetric subset of  $G$ . Then the **Group-Subgroup Pair Graph**  $\mathcal{G}(G, H, S)$  is the graph with vertex set  $G$  and edges

$$\begin{cases} (h, hs), (hs, h) & \forall h \in H, \forall s \in S - H, \\ (h, hs) & \forall h \in H, \forall s \in S \cap H \end{cases}$$

Equivalently, the group-subgroup pair graph  $\mathcal{G}(G, H, S)$  can be defined as

$$\mathcal{G}(G, H, S) = \bar{\mathcal{G}}(G, H, S_O) \oplus \mathcal{G}(H, S_H),$$

where  $S_O = S - H$ ,  $S_H = S \cap H$  and  $\oplus$  is the generalized edge sum operator, as defined in [8]. The graph  $\mathcal{G}(H, S_H)$  is a Cayley graph and the graph  $\bar{\mathcal{G}}(G, H, S_O)$  is the graph with vertices  $G$  and edges

$$\begin{cases} (h, hs) & \forall h \in H, \forall s \in S_O, \\ (x, xs^{-1}) & \forall x \in \bigcup_{s \in S_O} Hs, \forall s \in Hx \cap S_O. \end{cases}$$

According to the alternative definition, the group-subgroup pair graph  $\mathcal{G}(G, H, S)$  can be thought informally as the undirected graph consisting of an inner Cayley graph of  $H$  determined by  $S_H$  and an outer graph that connects vertices of  $H$  with vertices of  $\bigcup_{s \in S_O} Hs$ . The notation  $S_O$  and  $S_H$  is used throughout this paper with the same meaning as in the previous definition.

When  $G = H$ , the generating set is  $S = S \cap H$  and the corresponding graph is a Cayley graph, in other words,  $\mathcal{G}(G, H, S) = \mathcal{G}(G, S)$ . This justifies the claim that the group-subgroup pair graphs are a generalization of the Cayley graphs and the seemingly artificial condition of symmetry of  $S \cap H$ . When  $S$  is empty, we call the resulting pair graph  $\mathcal{G}(G, H, S)$  or Cayley graph  $\mathcal{G}(G, S)$  trivial.

**Example 2.2.** Let  $G = \mathbb{Z}/12\mathbb{Z}$ ,  $H = \{0, 3, 6, 9\}$ , and  $S = \{2, 4, 5, 7, 8\}$ , then  $S_H = \emptyset$  and  $S_O = S$ . The corresponding pair-graph  $\mathcal{G}(G, H, S)$  can be seen on figure 1.

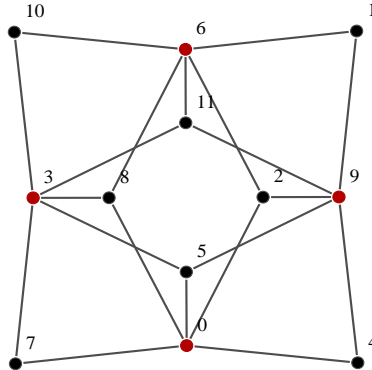


FIGURE 1. The pair graph  $\mathcal{G}(\mathbb{Z}/12\mathbb{Z}, H, S)$ .

**Example 2.3.** Let  $K = \mathbb{F}_{7^2}$ , and the prime field  $H = \mathbb{F}_7$  of  $K$  considered as a subgroup of the additive group  $K$ , then  $K$  is the direct sum of seven copies of  $H$ . Let  $\varphi$  be the norm map of  $K$  as a field extension of  $H$ , then if  $S = \varphi^{-1}(\{5, 6\})$ , we obtain the following pair-graph  $\mathcal{G}(K, H, S)$

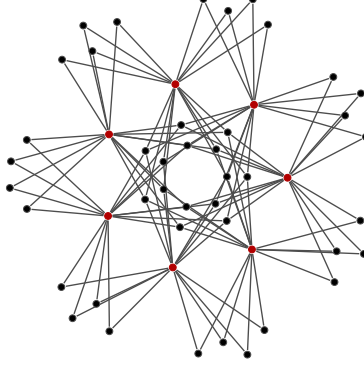


FIGURE 2. The pair-graph  $\mathcal{G}(\mathbb{F}_{7^2}, \mathbb{F}_7, S)$ .

This graph consists of vertices of degree 2, 4 and 16.

**Example 2.4.** Matrix groups over finite fields have been used in the construction of families of Ramanujan graphs. Set  $G = \text{GL}_2(\mathbb{F}_5)$ , where  $\mathbb{F}_5$  is the finite field of 5 elements and  $H = \text{SL}_2(\mathbb{F}_5)$ , then for a random subset  $S$  of 7 elements taken from the complement of  $H$  in  $G$ , we obtain the following pair-graph  $\mathcal{G}(G, H, S)$ .

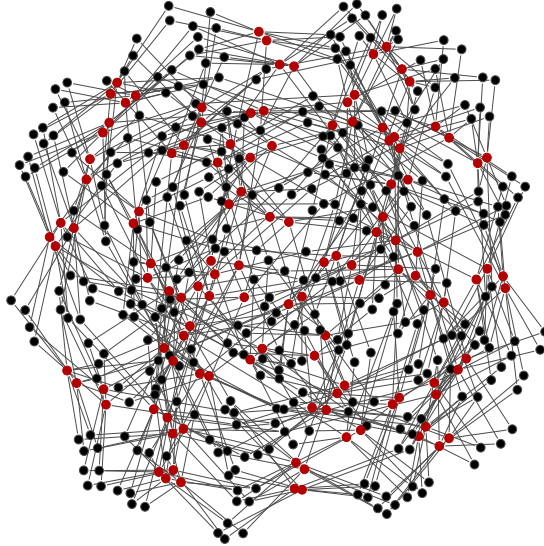


FIGURE 3. A pair-graph  $\mathcal{G}(\text{GL}_2(\mathbb{F}_5), \text{SL}_2(\mathbb{F}_5), S)$ .

This connected graph has 480 vertices with vertices of degrees 2, 3 and 7.

Note that none of the graphs of the preceding examples can be constructed as Cayley graphs.

*Remark 2.5.* We briefly mention another generalization of Cayley graphs for group  $G$  and subgroup  $H$ , the *Schreier Coset Graph*. For a symmetric subset  $S$  of  $G$ , the Schreier coset graph is defined as the graph with the set  $G/H$  of coset as vertices and where two cosets  $Hx$  and  $Hy$  are adjacent when there is an  $s \in S$  such that

$$Hxs = Hy.$$

A Schreier Coset graph can have multiples edges and loops (even when  $e \notin S$ ). This kind of graphs have been used for coset enumeration techniques. A detailed exposition can be found in [2].

For a given group  $G$ , the Schreier Coset graph is a Cayley graph when  $H = \{e\}$ , whereas the group-subgroup pair graph is a Cayley graph when  $H = G$ .

**2.2. Relation with group matrices.** As stated in the Introduction, one of the motivation for the group-subgroup pair graph comes from the extension of the group determinant for a group-subgroup pair, called *wreath determinant for group-subgroup pairs*. In this subsection we show how one can relate the adjacency matrix of a Cayley graph with the group matrix of the corresponding group; then, by extending the idea for the matrix used for the wreath determinant for group-subgroup pairs we obtain the rows corresponding to the subgroup on the adjacency matrix of a certain group-subgroup pair graph, which is enough to determine the complete adjacency graph.

For a group  $G = \{g_1, \dots, g_n\}$ , consider a polynomial ring  $R$  containing the indeterminates  $x_{g_i}$ , for  $g_i \in G$ , then the *group matrix* is a matrix  $\mathcal{M}(G, \phi)$  in  $\text{Mat}_{n,n}(R)$  defined by

$$\mathcal{M}(G, \phi)_{i,j} = x_{g_i^{-1}g_j}$$

for  $i, j \in [n]$  and where  $\phi : G \rightarrow [n]$  is an enumeration function for  $G$ , used implicitly. The determinant  $\Theta(G)$  of the group matrix is called *group determinant* of  $G$  and does not depend on the chosen enumeration of the elements of  $G$ . For  $i, j$  we have  $(g_i^{-1}g_j)^{-1} = g_j^{-1}g_i$ , therefore for any element  $x_g$ , the corresponding transpose element is  $x_{g^{-1}}$ .

Similarly, for a group  $G$  of order  $kn$ , and subgroup  $H$  of order  $n$  one can construct the matrix  $\mathcal{M}(G, H, \phi, \tau) \in \text{Mat}_{n,kn}(R)$  by

$$\mathcal{M}(G, H, \phi, \tau)_{i,j} = x_{h_i^{-1}g_j},$$

for  $h_i \in H$ ,  $g_j \in G$ ,  $i \in [n]$ ,  $j \in [nk]$  and where  $\phi : G \rightarrow [nk]$  and  $\tau : H \rightarrow [n]$  are enumerations functions for  $G$  and  $H$ . Considering only the columns corresponding to elements of  $H$  of the matrix  $\mathcal{M}(G, H, \phi, \tau)$  one obtains the group matrix of  $H$  with respect to the ordering  $\tau$ .

For a matrix  $A \in M_{n,kn}$ , the wreath determinant of  $A$  is defined as

$$\text{wrdet}_k(A) = \det^{-\frac{1}{k}}(A_{[k]}),$$

where  $A_{[k]}$  is the row  $k$ -flexing of the matrix  $A$  and  $\det^\alpha$  is the  $\alpha$ -determinant, for an extensive exposition of the wreath determinant and its properties the reader is referred to [7]. In the paper [5], the authors define the wreath determinant for the pair  $G$  and  $H$  by

$$\Theta(G, H, \phi, \tau) = \text{wrdet}_k(\mathcal{M}(G, H, \phi, \tau)).$$

In contrast with the ordinary group determinant, this wreath determinant for  $G$  and  $H$  depends on the enumeration functions  $\phi$  and  $\tau$ .

For a given group  $G$  and symmetric subset  $S$ , by evaluating the corresponding group matrix  $\mathcal{M}(G, \phi)$  by the rule  $x_s = 1$  for  $s \in S$  and  $x_g = 0$  for  $g \notin S$  one obtains a symmetric matrix. Since  $g_i^{-1}g_j = s$  implies  $g_i s = g_j$ , the corresponding matrix is the adjacency matrix of the Cayley graph  $\mathcal{G}(G, S)$ .

**Example 2.6.** Consider  $\mathfrak{S}_3$ , the symmetric group on three letters with the ordering  $\phi$  given by  $\mathfrak{S}_3 = \{e, (2, 3), (1, 2), (1, 2, 3), (1, 3, 2), (1, 3)\}$ , the group matrix is

$$\mathcal{M}(\mathfrak{S}_3, \phi) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_4 & x_3 & x_6 & x_5 \\ x_3 & x_5 & x_1 & x_6 & x_2 & x_4 \\ x_5 & x_3 & x_6 & x_1 & x_4 & x_2 \\ x_4 & x_6 & x_2 & x_5 & x_1 & x_3 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{pmatrix},$$

where  $x_i$  stands for  $x_{g_i}$ . Considering  $S = \{(1, 2), (1, 2, 3), (1, 3, 2)\}$  and evaluating in the way mentioned before we get

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

which can be verified to be the adjacency matrix of the Cayley graph  $\mathcal{G}(\mathfrak{S}_3, S)$ .

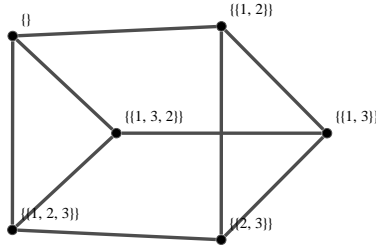


FIGURE 4. The Cayley graph  $\mathcal{G}(\mathfrak{S}_3, S)$ .

Likewise, for group  $G$ , subgroup  $H$  and subset  $S$  as in definition 2.1, by evaluating the group-subgroup matrix with the rule  $x_s = 1$  for  $s \in S$  and  $x_g = 0$  otherwise, we obtain a matrix with nonzero entries  $(i, j)$  when  $h_i^{-1}g_j = s \in S$ . In other words, there are ones in the matrix exactly when  $h_i s = g_j$ , which is the relation for the edges of the group-subgroup pair graph  $\mathcal{G}(G, H, S)$  in definition 2.1. The resulting matrix corresponds to the rows associated with the elements of  $H$  in the adjacency matrix of the pair-graph  $\mathcal{G}(G, H, S)$  and can be completed by symmetry to obtain the complete adjacency matrix.

**Example 2.7.** Let  $G = \mathbb{Z}/12\mathbb{Z}$ ,  $H = \{0, 3, 6, 9\}$ , and  $S = \{2, 4, 5, 7, 8\}$  as in example 2.2, the corresponding matrix with respect to natural orderings  $\phi$  and  $\tau$  is

$$\mathcal{M}(\mathbb{Z}/12\mathbb{Z}, H, \phi, \tau) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} \\ x_9 & x_{10} & x_{11} & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_0 & x_1 & x_2 \end{pmatrix}.$$

Then, evaluating as describe before

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which can be verified to correspond to the rows of the vertices of  $H$  of the adjacency matrix of the pair-graph  $\mathcal{G}(\mathbb{Z}/12\mathbb{Z}, H, S)$  of example 2.2.

Note that the group matrix can also be defined as  $(g_i g_j^{-1})$ . Using this definition the resulting Cayley graph is defined by left multiplication, the same is true for the group-subgroup matrix for the wreath determinant and the group-subgroup pair graph.

**2.3. Basic properties of pair graphs  $\mathcal{G}(G, H, S)$ .** An isolated vertex is one that is not connected to any other vertices. In contrast with nontrivial Cayley graphs, group-subgroup pair graphs may contain isolated vertices even when the generating subset is non empty. The following result characterizes the presence of isolated vertices in group-subgroup pair graphs.

**Proposition 2.8.** *i) The pair-graph  $\mathcal{G}(G, H, S)$  contains no isolated vertices if and only if  $S$  contains a representative for each coset of  $H$  on  $G$  different from  $He = H$ .*

*ii) The vertices  $H$  are isolated in  $\mathcal{G}(G, H, S)$  if and only if  $S$  is the empty set.*

*Proof.* Suppose  $S$  is not empty and contains a representative for each coset, then take  $x \in G - H$ , and  $s \in S$  the representative of  $Hx$ , then there is  $h \in H$  such that  $hs = x$ , and therefore  $x$  is connected to  $h$ . Conversely, if there are no isolated vertices, by the definition we must have  $\bigcup_{s \in S_0} Hs = G$ . The second statement follows directly from the definition.  $\square$

**Example 2.9.** Consider any group  $G$  of order  $n$  and  $H = \{e\}$ , then the graph  $\mathcal{G}(G, H, S)$  with  $S = G - H$  has no isolated vertices. In fact,  $\mathcal{G}(G, H, S)$  is a  $T_{n-1}$  star graph.

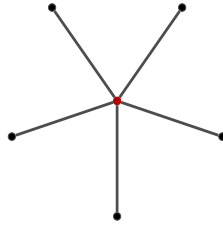


FIGURE 5. A  $\mathcal{G}(G, \{e\}, S)$  graph with  $|G| = 6$  and  $S = G - \{e\}$ .

The second part of the proposition shows the relation with Cayley graphs, as a Cayley graph is empty if and only if  $S$  is the empty set.

A graph in which all the vertices have the same degree is called a *regular graph*, more precisely, if the vertices have degree  $k$ , the graph is called a  *$k$ -regular graph*. An important property of a Cayley graph  $\mathcal{G}(G, S)$  is that it is  $|S|$ -regular. Example 2.2 shows that this is not true in general for  $\mathcal{G}(G, H, S)$ , but there is still uniformity on the degree of the vertices in each coset.

**Proposition 2.10.** *In a pair-graph  $\mathcal{G}(G, H, S)$ , all the vertices in the same coset have the same degree. Namely, the vertices in  $H$  have degree  $|S|$  and for  $x \notin H$  the degree of the vertices in the coset  $Hx$  is  $|S \cap Hx|$ .*

*Proof.* It is clear by the definition of the outer graph  $\bar{\mathcal{G}}(G, H, S_O)$  that any vertices  $x, y \in G - H$  in the same coset  $Hx$  have the same degree  $|Hx \cap S_O| = |Hy \cap S_O|$ . The vertices in  $H$  have degree  $|S|$  by construction.  $\square$

Returning to example 2.2,  $(H + 1) \cap S = \{4, 7\}$ ,  $(H + 2) \cap S = \{2, 5, 8\}$ , and the cardinality of these sets corresponds to the degree of the vertex in the respective cosets. We call those types of graphs, *multi-regular* or more precisely  $p_1, p_2, \dots, p_r$ -regular graphs, where  $p_i$  is the degree of the vertices on a given partition. The preceding discussion suggests that the structure of  $G/H$ , the set of cosets of  $H$  on  $G$ , is closely related to the structure of the graph.

**Corollary 2.11.** *Let  $G$  be a group and  $H$  a subgroup of index  $[G : H] = k + 1$ . For a subset  $S \subset G$  with  $S_H$  symmetric, consider the pair-graph  $\mathcal{G}(G, H, S)$ . If  $x_1, \dots, x_k$  is a set of representatives of the cosets of  $H$  incongruent to  $e$ , then for  $h \in H$ ,*

$$\deg(h) = |S| \geq \sum_{i=1}^k |Hx_i \cap S_O| = |S_O| = \sum_{i=1}^k \deg(x_i),$$

*with equality only when  $S_H = \emptyset$ . In particular, a nontrivial  $\mathcal{G}(G, H, S)$  is regular if and only if  $S_H = \emptyset$  and  $[G : H] = 2$ , or  $[G : H] = 1$ .*

*Proof.* The inequality follows from Proposition 2.10 and the fact that  $|S| \geq |S_O|$ , and the equality happens only when  $S_H = S \cap H = \emptyset$ . The *if* part of the proof follows directly from the inequality and the definitions. For the *only if* part, consider a  $j$ -regular  $\mathcal{G}(G, H, S)$  graph, then by the preceding proposition  $|S| = j$  and  $|Hx \cap S_O| = j$  for  $x \notin H$ . It follows from the inequality above that  $|S_O| = kj = k|S|$  and therefore  $k$  is necessarily 0 or 1. If  $[G : H] = 2$ , then  $|S_O| = |S|$ , so  $S = S_O$  and the case  $[G : H] = 1$  gives a Cayley graph.  $\square$

Note that in view of Proposition 2.10, we can consider  $\deg(Hx)$  as the degree of any of the elements of the coset. In that case, the above identity can be written as

$$\deg(H) \geq \sum_i \deg(Hx_i),$$

with equality happening only when  $S_H$  is empty.

### 3. GENERAL PROPERTIES OF PAIR-GRAPHS $\mathcal{G}(G, H, S)$

**3.1. Connectedness and connected components.** In this section we consider the connectedness for pair graphs  $\mathcal{G}(G, H, S)$ . Recall that a Cayley graph  $\mathcal{G}(G, S)$  is connected if and only if  $\langle S \rangle = G$ . We begin by considering the case  $S_H = S \cap H$  empty, in other words, none of the vertices of  $H$  are adjacent in  $\mathcal{G}(G, H, S)$ .



**Lemma 3.1.** *Let  $G$  be a group,  $H$  a subgroup and  $S \subset G$  subset with  $S_H$  symmetric. If  $S_H = \emptyset$  then in the pair-graph  $\mathcal{G}(G, H, S)$  the vertices of  $H$  are in the same connected component if and only if  $\langle H \cap (SS^{-1}) \rangle = H$ .*

*Proof.* If  $\langle H \cap (SS^{-1}) \rangle = H$ , then it suffices to prove that the identity  $e$  is connected to an arbitrary  $h \in H$ . For  $h \in H$ , we have  $h = s_1 s_2^{-1} \dots s_{n-1} s_n^{-1}$  with  $s_i s_{i+1}^{-1} \in H$ , then if we set  $h_1 = s_1 s_2^{-1} \dots s_{n-3} s_{n-2}^{-1}$ ,  $h_1$  is adjacent to  $h_1 s_{n-1} = x_1$  and  $h$  is adjacent to  $h s_n = h_1 s_{n-1} = x_1$  so  $h_1$  is connected to  $h$ . By repeating this process we conclude that  $e$  is connected to  $h$ .

On the other hand, if in the graph  $\mathcal{G}(G, H, S)$ , all the vertices of  $H$  are in the same connected component,  $h \in H$  is connected to  $e \in H$ . Since there are no direct connections between two elements of  $H$  or  $G - H$ , there must be path from  $e$  to  $h$  where every even vertex is an element of  $H$ , so we have a sequence  $h_0 = e, h_1, \dots, h_{n-1}, h_n = h$  of elements of  $H$ , where  $h_i$  and  $h_{i+1}$  is connected to  $x_i$   $i = 0, 2, \dots, n-1$  with  $x_i \in G - H$ . That is, we have a sequence of edges  $(h_0, x_0), (x_0, h_1) \dots (h_{n-1}, x_{n-1}), (x_{n-1}, h_n)$ , as shown in the following diagram.

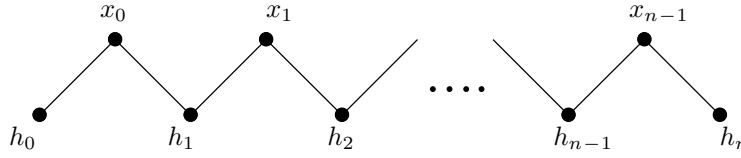


FIGURE 6. The path from  $h_0$  to  $h_n$ .

Then, for  $s_i \in S$ ,

$$\begin{aligned} x_0 &= h_0 s_0 & , & & x_0 &= h_1 s_1 \\ x_1 &= h_1 s_2 & , & & x_1 &= h_2 s_3 \\ & \vdots & & & \vdots & \\ x_{n-1} &= h_{n-1} s_{2n-2} & , & & x_{n-1} &= h_{n-1} s_{2n-1} \end{aligned}$$

thus,

$$\begin{aligned} s_0 &= h_0 s_0 = h_1 s_1 & h_1 &= s_0 s_1^{-1} \\ h_1 s_2 &= h_2 s_3 & h_2 &= h_1 s_2 s_3^{-1} \\ & \vdots & \Rightarrow & \vdots \\ h_{n-1} s_{n-2} &= h_n s_{2n-1} & h &= h_{n-1} s_{n-2} s_{n-1}^{-1}, \end{aligned}$$

it follows that  $s_i s_{i+1}^{-1} \in H$  and  $h \in \langle H \cap SS^{-1} \rangle$ .  $\square$

Note that since a group-subgroup pair graph may contain isolated vertices, the condition of the lemma alone is not sufficient for connectedness.

**Proposition 3.2.** *With the same notation as before, if  $S_H = \emptyset$  then the pair-graph  $\mathcal{G}(G, H, S)$  is connected if and only if  $\langle H \cap SS^{-1} \rangle = H$  and  $S$  contains representatives of all the cosets of  $H$  different from  $H$ .*

*Proof.* The result follows from part follows from Lemma 3.1, Proposition 2.8 and the observation that any vertex  $x \in G - H$  must be connected to some  $h \in H$  which is in turn connected to the identity  $e \in H$ .  $\square$

**Theorem 3.3.** A pair-graph  $\mathcal{G}(G, H, S)$  is connected if and only if

$$\langle H \cap (S_H \cup S_o S_o^{-1}) \rangle = H$$

and  $S$  contains representatives of all the cosets of  $H$  different from the coset  $H$ .

*Proof.* First we see that the vertices of  $H$  are in the same connected component if and only if  $\langle H \cap (S_H \cup S_o S_o^{-1}) \rangle = H$ . The proof of this fact is the same as that of Lemma 3.1 while considering that in the path from a  $e \in H$  to  $h \in H$  there may be edges connecting elements  $h_1, h_2$  from  $H$ , in such case we have  $h_2 = h_1 s_H$ , with  $s_H \in S_H$ . Then the result follows like in Proposition 3.2.  $\square$

**Example 3.4.** Let  $G = \mathbb{Z}/12\mathbb{Z}$ ,  $H \cong \mathbb{Z}/4\mathbb{Z}$ . Set  $S_1 = \{1, 7\}$  and  $S_2 = \{4, 5, 6, 10, 11\}$ , the corresponding group-subgroup pair graphs are the following.

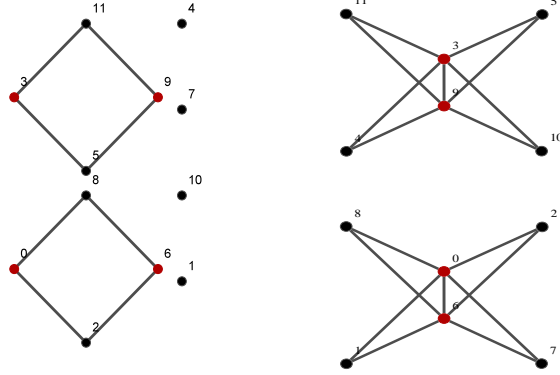


FIGURE 7. The pair-graphs  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$ .

Note that  $\langle H \cap S_1 S_1^{-1} \rangle = \{0, 6\}$  and  $\langle H \cap (S_{H_2} \cup S_{O_2} S_{O_2}^{-1}) \rangle = \{0, 6\}$ , so neither graph is connected, as can be seen in the diagrams. On the other hand, as there are no elements of the coset  $H + 5 = \{5, 8, 11, 2\}$  on  $S_1$ , all the vertices of that coset are isolated on  $\mathcal{G}(G, H, S_1)$ .

If a graph is not connected, the characterization of the connected components of the graph is desirable. For Cayley graphs, the connected component of the identity is the subgroup  $\langle S \rangle$ , and each of the cosets in  $G$  are the connected components of the graph. This identification is not identical for group-subgroup pair graphs as the connected component of the identity may include vertices from  $G - H$ . In particular, the subgroup  $\langle H \cap (S_H \cup S_o S_o^{-1}) \rangle$  of  $H$  contains the elements of  $H$  that are in the connected component of the identity and the cosets of this subgroup are the intersection of  $H$  with certain connected components of the graph.

**Proposition 3.5.** With the same notation as before, let  $U = \langle H \cap (S_H \cup S_o S_o^{-1}) \rangle$ , then the identity component  $\Gamma_e$  of  $\mathcal{G}(G, H, S)$  consists of the vertices  $U \cup (\bigcup_{s \in S_o} U s)$ . The remaining connected components of the pair-graph  $\mathcal{G}(G, H, S)$  are either of the type  $\Gamma_h = h \Gamma_e$  for  $h \in H$  or the type  $\{x\}$  for  $x \in G - H$ .

*Proof.* The first statement follows from the preceding discussion and the definition of the pair-graph  $\mathcal{G}(G, H, S)$ . Any path  $e, g_1, g_2, \dots, g_n$  from the identity  $e$  to  $g_n$  corresponds uniquely to a path  $h, h g_1, \dots, h g_n$  from  $h$  to  $h g_n$  so the connected

component of  $h \in H$  is  $\Gamma_h = h\Gamma_e$ . For  $x \in G - H$  if  $x$  is an isolated vertex its connected component is  $\{x\}$ , otherwise it is connected to an  $h \in H$  so its connected component is of the type  $\Gamma_h$ .  $\square$

A consequence of the above proposition is that an arbitrary connected component  $\Gamma$  of  $\mathcal{G}(G, H, S)$  has cardinality equal to  $\Gamma_e$  or 1. Moreover, for first case, we also have  $|\Gamma \cap H| = |\Gamma_e \cap H|$  and  $|\Gamma - H| = |\Gamma_e - H|$ .

For Cayley graphs, the number of connected components of the graph is the index  $[G : \langle S \rangle]$ . The existence of isolated vertices even for non empty generating sets makes the situation for pair-graphs slightly more complicated.

**Theorem 3.6.** *The number of connected components of  $\mathcal{G}(G, H, S)$  is*

$$[H : \langle H \cap (S_H \cup S_o S_o^{-1}) \rangle] + |G - H| - \left| \bigcup_{s \in S_o} Hs \right|.$$

*Proof.* By the preceding proposition, the first term in the formula is the number of connected components  $\Gamma_h$  that occur on  $H$ , the second and third terms count the number of isolated points in  $G - H$ , by Proposition 2.8. Since there are not connections between elements of  $G - H$ , this is the number of connected components of the graph.  $\square$

Proposition 3.5 and Theorem 3.6 completely characterize the connected components for the pair-graphs  $\mathcal{G}(G, H, S)$  for given group  $G$ , subgroup  $H$  and valid subset  $S \subset G$ .

**Example 3.7.** For the pair-graph of example 2.2 we have  $S = \{2, 4, 5, 7, 8\}$ , since  $h = 3 = 8 - 5 \in SS^{-1}$ ,  $H$  is generated by  $SS^{-1}$  and the first term is 1, the second term is 8 and since all the cosets are represented the last term is 8, and we get 1 connected component.

For the pair-graphs generated by  $S_1$  and  $S_2$  of example 3.4, in both cases the first term is 2, the next term is 8, and the final term is 4 for the graph generated by  $S_1$  and 8 for the graph generated by  $S_2$ , therefore  $\mathcal{G}(G, H, S_1)$  has 6 connected components and  $\mathcal{G}(G, H, S_2)$  has 2 connected components as confirmed visually in the diagrams.

**3.2. Vertex transitivity and group actions.** A graph is *vertex transitive* when for any pair of different vertices  $x$  and  $y$  there is graph automorphism  $\varphi$  such that  $\varphi(x) = y$ . Cayley graphs are naturally vertex transitive by means of left translations  $L_g$  with  $g \in G$ . Any vertex transitive graph must be regular, therefore by Proposition 2.10, we have the following result.

**Proposition 3.8.** *Nontrivial pair-graphs  $\mathcal{G}(G, H, S)$  are not vertex transitive when  $[G : H] \geq 3$ .*

However, we still see restricted transitivity when considering left translations by elements of  $H$  on  $\mathcal{G}(G, H, S)$ .

**Proposition 3.9.** *The left action of  $H$  in  $\mathcal{G}(G, H, S)$  is a graph automorphism. Moreover, for any  $h_1, h_2 \in H$ , the action is transitive. The same holds for any coset  $Hx$ ,  $x \in G$ .*

*Proof.* For  $h \in H$  and  $s \in S$ , the edge  $(h, hs)$  is in  $\mathcal{G}(G, H, S)$ . Now, for  $h' \in H$ , the images of  $h$ , and  $hs$  under the left action  $L_{h'}$  are  $h'h$  and  $h'hs$ , which are

adjacent in  $\mathcal{G}(G, H, S)$ . The map is a bijection, and has inverse  $L_{h'^{-1}}$  so it is a graph automorphism. For  $h_1, h_2 \in H$ , consider  $h' = h_1 h_2^{-1}$ , then  $L_{h'}(h_1) = h_2$  as required. For  $x_1, x_2 \in Hx$ , take  $h' \in H$  such that  $L_{h'}(x_1) = x_2$ .  $\square$

Note that the left action of arbitrary  $g \in G$  is not necessarily a graph automorphism. For instance, in example 1 the action of  $g = 1$  is not a graph endomorphism, as the image of 0 is 1, and  $\text{degree}(0) = 5$ , but  $\text{degree}(1) = 2$ .

Let  $\mathcal{L}(G)$  be the set of complex valued functions defined on  $G$ , and consider  $\lambda_H : H \rightarrow \text{GL}(\mathcal{L}(G))$  the left regular representation of  $H$  given by the action  $\lambda_H(h)f(x) = f(h^{-1}x)$  for  $h \in H$  and  $x \in G$ . Recall that for any graph with vertices on  $G$  we can associate the adjacency operator  $A$ , acting on  $\mathcal{L}(G)$  in the following way,

$$Af(x) = \sum_{x \sim y} f(y).$$

**Proposition 3.10.** *Let  $A$  be the adjacency matrix for a group-subgroup pair graph  $\mathcal{G}(G, H, S)$ , then for any  $h \in H$*

$$\lambda_H(h)A = A\lambda_H(h).$$

*Proof.* Note that for the pair-graph  $\mathcal{G}(G, H, S)$  and  $f \in \mathcal{L}(G)$ , the adjacency operator is given by

$$Af(y) = \begin{cases} \sum_{s \in S} f(ys) & \text{if } y \in H \\ \sum_{s \in S \cap H_y} f(ys^{-1}) & \text{if } y \in G - H \end{cases}$$

The result then follows from direct calculation by considering that for any  $x \in G$ ,  $x$  and  $h^{-1}x$  are in the same coset, in other words,  $Hx = Hh^{-1}x$  for any  $h \in H$ .  $\square$

For  $h \in H$ ,  $\lambda_H(h)$  is a permutation matrix for the elements of  $G$  corresponding to the left multiplication by  $h$ . By Theorem 15.2 of [1], a bijection  $\varphi$  of the vertices of a graph is a graph automorphism if  $M_\varphi A = A M_\varphi$ , where  $M_\varphi$  is the permutation matrix associated with  $\varphi$ ; therefore Proposition 3.10 is only a representation theoretic restatement of Proposition 3.9.

**3.3. The trivial eigenvalues of pair-graphs  $\mathcal{G}(G, H, S)$ .** The trivial eigenvalue of a  $k$ -regular graph is  $\mu = k$  and it corresponds to any constant eigenfunction  $f$  on the vertices of the graph. By extension, the trivial eigenvalue of a Cayley graph  $\mathcal{G}(G, S)$  is  $\mu = k$ . In this section we extend this notion to the group-subgroup pair graphs.

**Theorem 3.11.** *Let  $G$  be a group,  $H$  a subgroup of  $G$  of index  $[G : H] = k + 1$  with  $k \geq 1$ , and  $S \subset G$  a nonempty subset with  $S_H$  symmetric and  $|S_O| \neq 0$ . Consider  $e = x_0, x_1, \dots, x_k$  a set of representatives of the cosets of  $H$  in  $G$  and set  $S_i = S \cap Hx_i$ , for  $i \in [k]$ . Then*

$$\mu^\pm = \frac{|S_H| \pm \sqrt{|S_H|^2 + 4 \left( \sum_1^k |S_i|^2 \right)}}{2}$$

*are eigenvalues of the graph  $\mathcal{G}(G, H, S)$ . A corresponding eigenfunction is defined by*

$$f^\pm(y) = \begin{cases} \mu^\pm, & \text{if } y \in H \\ |S_i| & \text{if } y \in Hy_i, \quad i \in [k]. \end{cases}$$

*Proof.* Note that the numbers  $\mu^\pm$  satisfy

$$(\mu^\pm)^2 - |S_H|\mu^\pm - \sum_{i=1}^k |S_i|^2 = 0.$$

Then, for  $h \in H$ , we have

$$\begin{aligned} Af^\pm(h) &= \sum_{s \in S} f^\pm(hs) \\ &= \sum_{s \in S_H} f^\pm(hs) + \sum_{i=1}^k \sum_{s \in S_i} f^\pm(hs) \\ &= |S_H|\mu^\pm + \sum_{i=1}^k |S_i|^2 = (\mu^\pm)^2 \\ &= \mu^\pm f(h). \end{aligned}$$

Similarly, for  $x \in Hx_i$ ,  $i = 1, \dots, k$ ,

$$\begin{aligned} Af^\pm(x) &= \sum_{s \in S_i} f(xs^{-1}) \\ &= \mu^\pm |S_i| \\ &= \mu^\pm f^\pm(x). \end{aligned} \quad \square$$

Note that for the case  $|S_O| = 0$ ,  $\mu^+$  is an eigenvalue with corresponding eigenvector  $f^+$  as defined in the above Theorem, but  $f^- \equiv 0$  so it is not an eigenfunction.

**Proposition 3.12.** *With the same notation as before, the eigenvalue  $\mu^+$  is the largest eigenvalue of the graph  $\mathcal{G}(G, H, S)$  with multiplicity  $[H, \langle H \cap (S_H \cup S_O S_O^{-1}) \rangle]$ .*

*Proof.* First, consider the case of a connected pair-graph  $\mathcal{G}(G, H, S)$ , in particular  $|S_i| \neq 0$  for all  $i \in [k]$  and the eigenfunction  $f^+$  takes only positive values. By Thm. 8.1.4 and Cor. 8.1.5 of [8], any eigenfunction that only takes nonzero values of the same sign corresponds to the largest eigenvalue, which has multiplicity 1. The proposed eigenfunction satisfies this condition, therefore corresponds to the largest eigenvalue and it is an eigenvalue of multiplicity one, so the statement of the proposition follows.

For the remaining case, let  $h \in H$  and consider the connected component  $\Gamma_h$  as a subgraph of  $\mathcal{G}(G, H, S)$  and note that  $f^+|_{\Gamma_h}$  is an eigenfunction of  $\Gamma_h$  with eigenvalue  $\mu^+$ . Then by the same argument as before,  $\mu^+$  is the largest eigenvalue of  $\Gamma_h$  with multiplicity 1. Now, it is well known that the characteristic polynomial  $p_{\mathcal{G}}(x)$  of the graph  $\mathcal{G} = \mathcal{G}(G, H, S)$  is the product of the characteristic polynomials of its connected components,

$$p_{\mathcal{G}}(x) = p_{\Gamma_{h_1}}(x) p_{\Gamma_{h_2}}(x) \dots p_{\Gamma_{h_r}}(x) x^l,$$

where  $r$  is the number of connected components of  $\mathcal{G}(G, H, S)$  containing elements of  $H$  and  $l$  the number of isolated vertices. Moreover, since  $\mu^\pm$  is the largest root of  $p_{\Gamma_{h_i}}(x)$  for each  $h_i \in H$ , then is the largest root of  $p_{\mathcal{G}}(x)$  and therefore the largest

eigenvalue of  $\mathcal{G}(G, H, S)$ . Furthermore, since  $\mu^+$  is a simple eigenvalue for each of the subgraphs  $\Gamma_{h_i}$ , then by Theorem 3.6  $\mu^+$  is an eigenvalue of  $\mathcal{G}(G, H, S)$  with multiplicity equal to  $[H, \langle H \cap (S_H \cup S_O S_O^{-1}) \rangle]$ .  $\square$

From the decomposition of the characteristic polynomial of the adjacency matrix, we can obtain a bound for the multiplicity of the eigenvalue  $\mu = 0$ .

**Proposition 3.13.** *With the same notation as before,  $\mu = 0$  is an eigenvalue of multiplicity at least*

$$|G| - |H| - \min\left(\left|\bigcup_{s \in S_O} Hs\right|, |H|\right)$$

*of the nontrivial pair-graph  $\mathcal{G}(G, H, S)$ .*

*Proof.* First, suppose that the pair-graph  $\mathcal{G}(G, H, S)$  contains isolated vertices, then from the decomposition of the characteristic polynomial of the adjacency matrix we have that  $\mu = 0$  is an eigenvalue with multiplicity at least the number of isolated components, by 3.6 this number is  $|G| - |H| - |\bigcup_{s \in S_O} Hs|$ . For a general pair-graph  $\mathcal{G}(G, H, S)$ , consider an eigenfunction  $f$  associated with the eigenvalue  $\mu = 0$  with  $f(h) = 0$  for  $h \in H$ , then  $f$  must satisfy the  $|H|$  linear equations

$$\sum_{s \in S_O} f(hs) = 0.$$

The matrix  $B$  corresponding to this system is a  $|H| \times |G - H|$  matrix, then from elementary linear algebra it holds that  $|H|$  is an upper bound for the rank of  $B$ . Therefore the kernel of  $B$  has dimension at least  $|G| - 2|H|$ , so the multiplicity of the eigenvalue  $\mu = 0$  is at least  $|G| - 2|H|$ . The result follows from considering the two cases at the same time.  $\square$

**Example 3.14.** The pair-graph in example 2.3, has  $|S_O| = |S| = 16$ , with four cosets of degree 2 and two cosets of degree 4, therefore the trivial eigenvalues are  $\pm 4\sqrt{3}$ . The pair-graph in example 2.4 has  $|S_O| = |S| = 7$ , with two cosets of degree 2 and one coset of degree 3, then  $\pm\sqrt{17}$  are the corresponding trivial eigenvalues. The pair-graph in example 3.16 has  $|S_H| = 2$ ,  $|S_1| = |S_2| = 2$ , therefore the trivial eigenvalues are  $\mu^+ = 4$  and  $\mu^- = -2$ .

**3.4. Bipartite pair-graphs  $\mathcal{G}(G, H, S)$ .** A graph is *bipartite* when there is a bipartition  $V_+, V_-$  of the vertices such that any pair of vertices in the same subset are not adjacent. For a group  $G$  and symmetric subset  $S$ , if there is an homomorphism  $\chi : G \rightarrow \{-1, 1\}$ , such that  $\chi(S) = \{-1\}$  then the Cayley graph  $\mathcal{G}(G, S)$  is bipartite, this condition is also necessary when the graph is connected.

**Proposition 3.15.** *If a group homomorphism  $\chi : G \rightarrow \{-1, 1\}$ , such that  $\chi(S) = \{-1\}$  exists, then the graph  $\mathcal{G}(G, H, S)$  is bipartite. The converse is true if  $\mathcal{G}(G, H, S)$  is connected and  $S$  is symmetric.*

The proof of this proposition is the same as the one for Cayley graphs, see for example chapter 4 of [3]. Note that if such a homomorphism exists then  $\chi(S) = \{-1\}$ , so the elements of  $S$  and  $S^{-1}$  would be on the same partition.

**Example 3.16.** Let  $G = A_4$ , the alternating group of 4 letters,  $H$ , the Klein four group embedded as a subgroup of  $G$ , and  $S = \{(1, 2)(3, 4), (1, 4)(2, 3), (1, 2, 3),$

$(1, 4, 3), (2, 3, 4), (2, 4, 3)\}$ , using cycle notation. Observe that  $(1, 3, 2)$ , the inverse of  $(1, 2, 3)$ , is not contained in  $S$ . The resulting  $\mathcal{G}(G, H, S)$  is a bipartite graph.

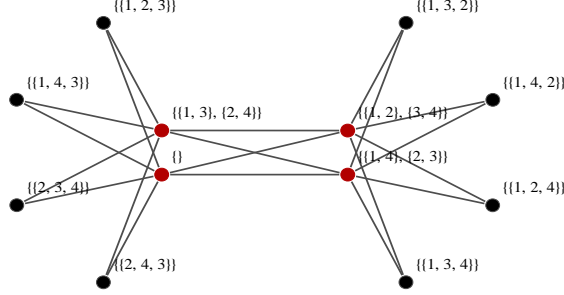


FIGURE 8. The bipartite pair-graph  $\mathcal{G}(G, H, S)$ .

Any homomorphism  $\chi : G \rightarrow \{-1, 1\}$  with  $\chi(S) = \{-1\}$ , would have  $\chi(S^{-1}) = \{-1\}$ , but in this case  $(1, 3, 2) = (1, 2)(3, 4) \cdot (1, 4, 3)$ , therefore there are no homomorphisms that satisfy the conditions.

For the case  $S \cap H = \emptyset$ , the sets  $H$  and  $G - H$  are a bipartition of  $\mathcal{G}(G, H, S)$ , so the nontrivial pair-graphs  $\mathcal{G}(G, H, S)$  with  $S \cap H = \emptyset$  are bipartite. An example can be seen in Figure 1.

#### 4. REGULAR PAIR-GRAPHS $\mathcal{G}(G, H, S)$

From the results of Section 2, a group-subgroup pair graph  $\mathcal{G}(G, H, S)$  is regular when  $H = G$ , or when  $H$  is a subgroup of order 2 and  $S_H$  is empty. In this section we restrict to the latter case, where the resulting pair-graph is also bipartite.

**Proposition 4.1.** *Let  $G, H$  and  $S$  be as described above. If  $S$  is a symmetric set, the resulting  $\mathcal{G}(G, H, S)$  is a Cayley graph. Namely,  $\mathcal{G}(G, H, S) = \mathcal{G}(G, S)$ .*

*Proof.* The conditions imply that  $S_O = S$ , then by the alternative definition of the pair-graph the edges are given by

$$(h, hs), \forall h \in H, \forall s \in S \quad \text{and} \quad (x, xs^{-1}), \forall x \in (G - H), \forall s \in S.$$

Since  $S$  is symmetric one can simply write  $(x, xs)$ ,  $\forall x \in G, \forall s \in S$ , which is the definition of Cayley graph.  $\square$

Proposition 4.1 states another way in which the graph  $\mathcal{G}(G, H, S)$  can be seen as a generalization of certain class of bipartite Cayley graphs. As an example, for  $p, q$  prime numbers, when  $p$  is not a square modulo  $q$ , the  $X^{p,q}$  Ramanujan graphs from Lubotzky, Phillips and Sarnak [9] can be identified with pair-graphs  $\mathcal{G}(\text{PGL}_2(q), \text{PSL}_2(q), S_{p,q})$ .

**4.1. Generating sets and actions.** In this section and the next one we fix a group  $G$  and subgroup  $H$  of order  $n$  and index 2. Let  $\mathcal{P}$  denote the power set of  $G - H$ . In this section we consider transformations on the generating sets that result in isomorphic group-subgroup pair graphs.

**Proposition 4.2.** *Let  $S_1 \in \mathcal{P}$ , if  $S_2 = R_{h'}(S_1)$ , where  $R_{h'}$  is the right translation by  $h' \in H$ , then  $\mathcal{G}(G, H, S_1) \cong \mathcal{G}(G, H, S_2)$ .*

*Proof.* For  $h' \in H$  consider the map  $\varphi_{h'} : G \rightarrow G$ , defined as  $\varphi_{h'}(x) = xh'$  for  $x \in G - H$  and  $\varphi(h) = h$  for  $h \in H$ . The map is clearly bijective, then for  $h \in H, s \in S_1$ , we have

$$\begin{aligned} (h, hs) &\mapsto (h, hsh') = (h, hs'), \\ (x, xs^{-1}) &\mapsto (xh', xs^{-1}) = (xh', xh's'^{-1}), \end{aligned}$$

with  $s' = sh' \in S_2$ , therefore is an graph homomorphism with inverse  $\varphi_{h'^{-1}}$  and the graphs  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$  are isomorphic.  $\square$

**Proposition 4.3.** *Let  $S_1 \in \mathcal{P}$ , if  $S_2 = \psi(S_1)$ , where  $\psi$  is a group automorphism of  $G$ , then the pairs graphs  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$  are isomorphic.*

*Proof.* Since  $[G : H] = 2$ ,  $H$  is invariant under  $\psi$ , then for any  $h \in H, x \in G - H, s \in S_1$ ,

$$\begin{aligned} (h, hs) &\mapsto (\psi(h), \psi(h)\psi(s)) = (\psi(h), \psi(h)s'), \\ (x, xs^{-1}) &\mapsto (\psi(x), \psi(x)s'^{-1}), \end{aligned}$$

with  $s' = \psi(s) \in S_2$ , therefore  $\psi$  is a graph isomorphism with inverse  $\psi^{-1}$ .  $\square$

Any orbit on  $\mathcal{P}$  of right actions  $R_h$  by elements of  $H$  and group isomorphisms  $\psi$  of  $G$  consists of sets that generate isomorphic graphs.

**4.2. Spectra of pair-graphs  $\mathcal{G}(G, H, S)$ .** As mentioned in the beginning of the section, under the current assumptions the pair-graphs  $\mathcal{G}(G, H, S)$  are regular bipartite. The spectrum of these graphs is symmetric about 0 and its largest eigenvalue is the trivial eigenvalue  $\mu_0 = |S|$ . Moreover, any eigenvalue  $\mu$  satisfies  $|\mu| \leq |S|$ . In this case the adjacency operator for the pair-graph  $\mathcal{G}(G, H, S)$  is given by

$$Af(y) = \begin{cases} \sum_{s \in S} f(ys) & \text{if } y \in H \\ \sum_{s \in S} f(ys^{-1}) & \text{if } y \in G - H \end{cases}$$

**Proposition 4.4.** *If  $S_1 \cup S_2 = G - H$  and  $S_1 \cap S_2 = \emptyset$ , then the adjacency operator of  $\mathcal{G}(G, H, G - H)$  is the sum of the adjacency operators of  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$ .*

*Proof.* Follows from the definition

$$\begin{aligned} \sum_{s \in G-H} f(hs) &= \sum_{s \in S_1} f(hs) + \sum_{s \in S_2} f(hs), \\ \sum_{s \in G-H} f(xs^{-1}) &= \sum_{s \in S_1} f(xs^{-1}) + \sum_{s \in S_2} f(xs^{-1}). \end{aligned} \quad \square$$

For a graph  $\mathcal{G} = (V, E)$ , the complement graph is  $\bar{\mathcal{G}} = (V, \bar{E})$ , where  $(x, y) \in \bar{E}$ , if and only if  $(x, y) \notin E$  for  $x \neq y$ . The eigenvector and eigenvalues of a  $k$ -regular graph and its complement can be related, see for example Lemma 8.5.1 of [4]. We present a relation between the of eigenvalues and eigenvectors for pair-graphs  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$  with  $S_1 \cup S_2 = G - H$  and  $S_1 \cap S_2 = \emptyset$ .

First, assume that  $|S_1| = k$ ,  $|S_2| = n - k$ , then any constant function  $f$  is an eigenfunction of both of the pair-graphs  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$  corresponding to  $\mu_1 = k$  and  $\lambda_1 = n - k$  respectively. Similarly, for  $c \in \mathbb{C}$ , the function  $f = c(\delta_H - \delta_{G-H})$  is an eigenfunction of  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$  corresponding to  $\mu_{2n} = -k$  and  $\lambda_{2n} = n - k$ .



**Theorem 4.5.** *Let  $S_1$  and  $S_2$  as before with  $|S_1| = k, |S_2| = n - k$  for numbers  $0 < k < n$ , then*

- (i) *if  $\mathcal{G}(G, H, S_1)$  is not connected, then there are independent eigenfunctions  $f$  and  $g$  of  $\mathcal{G}(G, H, S_1)$  associated to  $\mu = \pm k$  such that  $f - g$  is an eigenfunction of  $\mathcal{G}(G, H, S_2)$  corresponding to the eigenvalue  $-\mu$ .*
- (ii) *if  $f$  is an eigenfunction of  $\mathcal{G}(G, H, S_1)$  associated with an eigenvalue  $\mu \neq \pm k$ , then  $f$  is an eigenfunction of  $\mathcal{G}(G, H, S_2)$  corresponding to the eigenvalue  $-\mu$ .*

*Proof.* Denote the adjacency graph operator of graphs  $\mathcal{G}(G, H, S_1)$ ,  $\mathcal{G}(G, H, S_2)$ ,  $\mathcal{G}(G, H, G - H)$  as  $A, B, C$  respectively, so that  $C = A + B$ .

(i) For a connected components  $\Gamma$  of  $\mathcal{G}(G, H, S_1)$ , consider the function  $f = \delta_\Gamma$  or  $f = \delta_{\Gamma \cap H} - \delta_{\Gamma - H}$ , which can be verified to be eigenfunctions corresponding to  $\mu = k$  and  $\mu = -k$ , respectively. We can similarly define  $g$  with respect to a different connected component  $\Omega$  of  $\mathcal{G}(G, H, S_1)$ . Note that there are no isolated vertices on the graph, therefore by Proposition 3.5, all the connected components have the same cardinality, in particular  $|\Gamma \cap H| = |\Omega \cap H|$  and  $|\Gamma - H| = |\Omega - H|$ . Then, we have

$$\begin{aligned} Cf(h) &= \sum_{x \in G-H} f(x) = \sum_{x \in \kappa-H} f(x) && \text{for } h \in H, \\ Cf(x) &= \sum_{h \in H} f(h) = \sum_{h \in \kappa \cap H} f(h) && \text{for } x \in G - H. \end{aligned}$$

It follows that  $Cf = \delta_H |\Gamma - H| + \delta_{G-H} |\Gamma \cap H|$  or  $Cf = -\delta_H |\Gamma - H| + \delta_{G-H} |\Gamma \cap H|$  depending on the whether  $\mu = k$  or  $\mu = -k$ . By the preceding discussion, we have  $C(f - g) = 0$  and

$$B(f - g) = (C - A)(f - g) = C(f - g) - A(f - g) = -\mu(f - g).$$

(iii) For an eigenfunction  $f$  of  $A$  corresponding to an eigenvalue  $\lambda \neq \pm k$ , we have

$$Bf(x) = (C - A)f(x) = Cf(x) - \mu f(x),$$

therefore it suffices to prove  $Cf(x) = 0$ . In other words, it suffices to prove the two equalities

$$\begin{aligned} \sum_{h \in H} f(h) &= 0, \\ \sum_{x \in G-H} f(x) &= 0. \end{aligned}$$

First, note that for fixed  $s \in S_1$  we have

$$\sum_{h \in H} f(hs) = \sum_{x \in G-H} f(x),$$

then, by summing over all elements of  $S_1$ ,

$$\sum_{s \in S} \sum_{h \in H} f(hs) = \sum_{s \in S} \sum_{x \in G-H} f(x) = k \left( \sum_{x \in G-H} f(x) \right).$$

Since  $f$  is an eigenfunction associated with  $\mu$ , we have

$$\sum_{s \in S} \sum_{h \in H} f(hs) = \sum_{h \in H} \sum_{s \in S} f(hs) = \mu \left( \sum_{h \in H} f(h) \right),$$

therefore we have

$$(2) \quad k \left( \sum_{x \in G-H} f(x) \right) = \mu \left( \sum_{h \in H} f(h) \right).$$

Similarly, by fixing  $s \in S_1$  and using the equation

$$\sum_{x \in G-H} f(xs^{-1}) = \sum_{h \in H} f(h),$$

we obtain the relation

$$(3) \quad k \left( \sum_{h \in H} f(h) \right) = \mu \left( \sum_{x \in G-H} f(x) \right).$$

Finally, combining (2) and (3), we obtain the equations

$$\begin{aligned} \sum_{h \in H} f(h) &= \frac{\mu^2}{k^2} \sum_{h \in H} f(h), \\ \sum_{x \in G-H} f(x) &= \frac{\mu^2}{k^2} \sum_{x \in G-H} f(x). \end{aligned}$$

Since  $|\mu| \neq k$ , we conclude

$$\begin{aligned} \sum_{h \in H} f(h) &= 0, \\ \sum_{x \in G-H} f(x) &= 0, \end{aligned}$$

and the proof is complete.  $\square$

Recall that when  $\mathcal{G}(G, H, S_1)$  is not connected, by Theorem 3.11 the eigenvalue  $\mu = k$  has multiplicity equal to the number  $c$  of connected components of  $\mathcal{G}(G, H, S_1)$ , then by (i) the graph  $\mathcal{G}(G, H, S_2)$  also has the eigenvalue  $m = k$  with multiplicity  $c - 1$ . We can reformulate the result as follows.

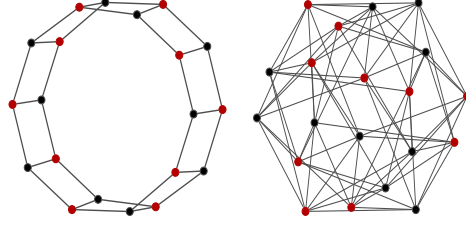
**Corollary 4.6.** *For group  $G$  and subgroup  $H$  of index 2 and order  $n$ . Let  $S \subset G - H$  with  $|S| = k$  and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{2n}$  the spectrum of the pair-graph  $\mathcal{G}(G, H, S)$ . Then there is a  $(n - k)$ -regular pair-graph  $\mathcal{G}(G, H, S')$  with spectrum  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{2n}$  such that*

$$\lambda_i = \mu_i$$

for  $i \neq 1, 2n$ .

**Example 4.7.** For  $G = \mathbb{Z}/20\mathbb{Z}$ ,  $H = \mathbb{Z}/10\mathbb{Z}$ , and  $S_1 = \{3, 5, 7\}$ ,  $S_2 = \{1, 3, 5, 13, 15, 17, 19\}$  we have the  $\mathcal{G}(G, H, S_1)$  and  $\mathcal{G}(G, H, S_2)$  pair-graphs shown in figure 9. Table 1 contains the values of the positive eigenvalues for both of the graphs. Since both graphs are bipartite the remaining eigenvalues correspond to the negatives of the ones shown.

Note that  $S_1 \cup S_2 \neq G - H = \{1, 3, 5, \dots, 19\}$ , but  $S' = R_4(S_1) = \{7, 9, 11\}$  is such that  $\mathcal{G}(G, H, S_1) \cong \mathcal{G}(G, H, S')$  by Proposition 4.2, and  $S' \cup S_2 = G - H$ .

FIGURE 9. Two  $\mathcal{G}(G, H, S)$  with the same nontrivial spectrum.

$\lambda_i$	$\mu_i$
3	7
$\frac{1}{2}(3 + \sqrt{5})$	$\frac{1}{2}(3 + \sqrt{5})$
$\frac{1}{2}(3 - \sqrt{5})$	$\frac{1}{2}(3 - \sqrt{5})$
$\frac{1}{2}(1 + \sqrt{5})$	$\frac{1}{2}(1 + \sqrt{5})$
$\frac{1}{2}(1 - \sqrt{5})$	$\frac{1}{2}(1 - \sqrt{5})$
1	1
$\frac{1}{2}(-1 + \sqrt{5})$	$\frac{1}{2}(-1 + \sqrt{5})$
$\frac{1}{2}(-1 - \sqrt{5})$	$\frac{1}{2}(-1 - \sqrt{5})$
$\frac{1}{2}(3 - \sqrt{5})$	$\frac{1}{2}(3 - \sqrt{5})$
$\frac{1}{2}(3 + \sqrt{5})$	$\frac{1}{2}(3 + \sqrt{5})$

TABLE 1. The positive values of the spectrum of the previous graphs.

*Remark 4.8.* Corollary 4.6 defines a symmetric relation between the nontrivial spectrum of graphs for complementary choices of  $S$  and the previous example shows that for a  $k$ -regular graph  $\mathcal{G}(G, H, S)$ , there can be more than one  $(n - k)$ -regular  $\mathcal{G}(G, H, S')$  graphs with the same nontrivial spectrum. In fact, if we find one, by using Proposition 4.2 with right actions of  $H$  we can obtain families of graphs with the same nontrivial spectrum. In order for the relation to be one to one, we need to consider equivalence classes of  $\mathcal{G}(G, H, S)$  graphs under graph isomorphism. An unsettled question is whether the number of equivalence classes of  $k$ -regular  $\mathcal{G}(G, H, S)$  is the same as that of  $(n - k)$ -regular ones for fixed group  $G$  and subgroup  $H$  of index 2.

Recall that a Ramanujan graph is a connected  $k$ -regular graph such that for every eigenvalue  $\mu$  different from  $\pm k$ , one has

$$|\mu| \leq 2\sqrt{k-1}.$$

**Corollary 4.9.** *With the same assumptions as before. If the pair graph  $\mathcal{G}(G, H, S)$  is connected and*

$$|S| \geq n + 2 - 2\sqrt{n},$$

*then it is a bipartite Ramanujan graph.*

*Proof.* Due to Theorem 4.5 any  $k$ -regular pair-graph  $\mathcal{G}(G, H, S)$  has nontrivial eigenvalues  $\mu$  satisfying  $|\mu| \leq \min\{k, n - k\}$ . Also, the pair-graph  $\mathcal{G}(G, H, S)$  is a Ramanujan graph when said trivial eigenvalues satisfy  $|\mu| \leq 2\sqrt{k-1}$ . Considering the two inequalities, it follows that all  $k$ -regular pair-graphs  $\mathcal{G}(G, H, S)$  with  $k \leq 2$  or  $k \geq n + 2 - 2\sqrt{n}$  are Ramanujan graphs.  $\square$

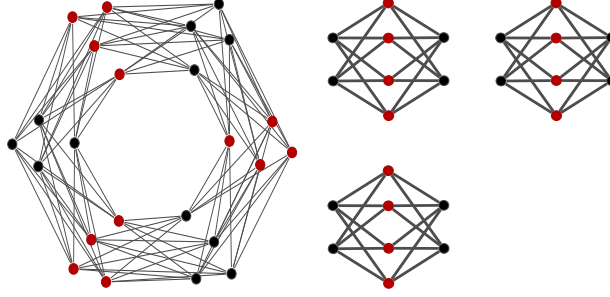
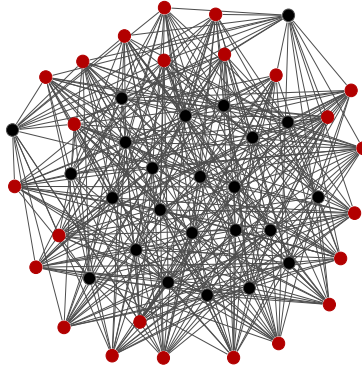


FIGURE 10. A Ramanujan pair-graph and its corresponding pair-graph.

*Remark 4.10.* Note that the condition is not necessary. Also, since the bound  $n + 2 - 2\sqrt{n}$  grows with the size of  $n$  it is not immediately useful for finding families of Ramanujan graphs, but it may be useful for the construction and verification of Ramanujan graphs.

**Example 4.11.** Let  $G = \mathfrak{S}_4$ ,  $H = A_4$ . The set  $S = \{(1, 2), (1, 3), (2, 4), (3, 4), (1, 2, 3, 4), (1, 3, 2, 4), (1, 4, 2, 3), (1, 4, 3, 2)\}$  is such that  $|S| = 8$  satisfy the bound of the corollary, so the corresponding  $\mathcal{G}(G, H, S)$  graph is Ramanujan. Its spectrum consist of  $\pm 8$  with multiplicity 1,  $\pm 4$  with multiplicity 2 and 0 with multiplicity 18. Note that it contains the eigenvalues  $\pm 4$  and as expected by Theorem 4.5, the corresponding 4-regular graph, generated by  $S' = \{(1, 2), (3, 4), (1, 3, 2, 4), (1, 4, 2, 3)\}$  has 3 connected components. Both of the pair-graphs are shown in figure 10.

**Example 4.12.** Let  $G = \text{GL}_2(\mathbb{F}_3)$  and  $H = \text{SL}_2(\mathbb{F}_3)$ . Then  $|G| = 48$  and  $|H| = 24$ , in this case by Corollary 4.9, for any subset  $S$  with  $|S| \geq 17$ , the resulting pair-graph  $\mathcal{G}(G, H, S)$  graph is Ramanujan. For a random choice of  $S$  we obtain the 17-regular Ramanujan pair graph  $\mathcal{G}(G, H, S)$  shown in figure 11.

FIGURE 11. The Ramanujan pair-graph  $\mathcal{G}(G, H, S)$ .

In this case, the pair-graph  $\mathcal{G}(G, H, (G - H) - S)$  of figure 12, generated by the complement of  $S$  in  $G - H$ , is a 7-regular Ramanujan graph. This shows that

the result of Corollary 4.9 is not a necessary condition, as mentioned in the above remark.

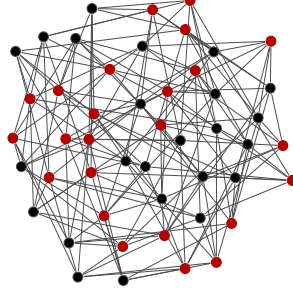


FIGURE 12. The Ramanujan pair-graph  $\mathcal{G}(G, H, (G - H) - S)$ .

As mentioned in the Introduction, Ramanujan graphs are graphs with good connectivity (as measured by the isoperimetric constant of the graph). Since the addition of edges to a graph results in a graph with greater or equal connectivity, it is expected for graphs with large amount of vertices to have good connectivity. This can be confirmed in the above examples, as the Ramanujan pair-graphs  $\mathcal{G}(G, H, S)$  resulting from using the bound provided by Corollary 4.9, for group  $G$  and subgroup  $H$  of order  $n \geq 4$ , have generating set  $S$  satisfying  $|S| \geq n/2$ . Then, Corollary 4.9 can also be stated as follows: The set of graphs

$$\{\mathcal{G}(G, H, S) | S \subset G - H, |S| \geq m\}$$

consists only of Ramanujan graphs when  $m = n + 2 - 2\sqrt{n}$ , where  $H$  is a subgroup of order  $n$  and  $[G : H] = 2$ .

## 5. SUMMARY

The general properties of group-subgroup pair graphs according to the choice of  $G, H$  and  $S = \emptyset$  are shown in diagram 5. For arbitrary pair-graph  $\mathcal{G}(G, H, S)$  a set of trivial eigenvalues can be determined including the largest eigenvalue of the graph. According to the index of the subgroup the pair-graph can be regular (index 1 or 2) or never be regular. If the generating set is contained in complement of the subgroup, the pair-graph is bipartite or multipartite. The pair-graphs are Cayley graphs in two cases: when the subgroup and the group are equal and when the subgroup is of order 2 and the graphs is regular bipartite with symmetric generating set. The symmetry on the generating set also makes easier to verify if the given pair-graph is bipartite. These observations suggest that further research on the matter may be started with graphs with symmetric subsets before considering the general case.

## 6. ACKNOWLEDGMENTS

The author was supported by the Japanese Government (MONBUKAGAKUSHO: MEXT) Scholarship.

The author would like to express his gratitude to Professor Masato Wakayama for providing the motivation for the present work and for the invaluable discussions

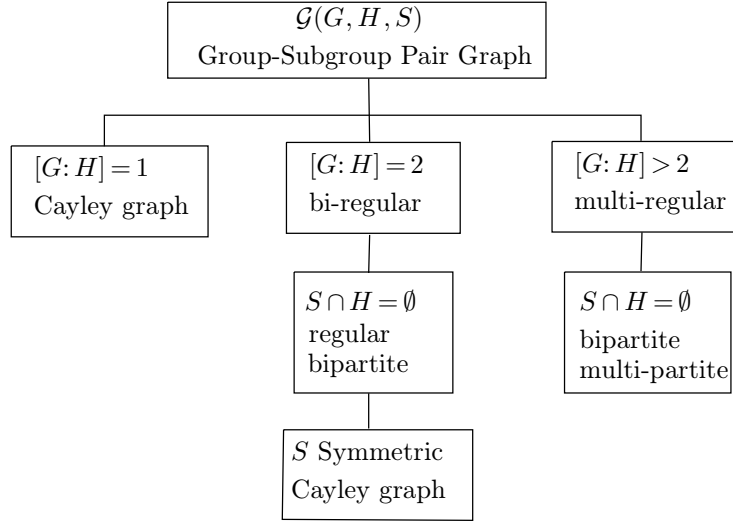


FIGURE 13. Characteristics of group-subgroup pair graphs.

and support. The author would also like to thank Professors Yoshinori Yamasaki, Miki Hirano and Kazufumi Kimoto for their valuable comments and discussion, regarding the definition of the Group-Subgroup pair graph during the event “Zeta Functions in Okinawa 2013”; and Mr. Hiroto Inoue for his valuable suggestions on the statements and proofs of some of the results.

The computations and diagrams were elaborated using Mathematica 9.0 Student Edition. The source files for the diagrams can be downloaded from

<http://www2.math.kyushu-u.ac.jp/~ma213054/files/figures.nb>.

#### REFERENCES

1. Norman Biggs. *Algebraic Graph Theory*. Cambridge Mathematical Library, 1996.
2. H. S. M. Coxeter and W. O. J Moser. *Generators and Relations for Discrete Groups*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer Berlin Heidelberg, 1972.
3. Guiliiana Davidoff, Peter Sarnak and Alain Valette. *Elementary Number Theory, Group Theory and Ramanujan Graphs*. London Mathematical Society student texts; 55, 2003.
4. Chris Godsil and Gordon Royle. *Algebraic Graph Theory*. Graduate Texts in Mathematics; 207, 2001.
5. Kei Hamamoto, Kazufumi Kimoto, Kazutoshi Tachibana and Masato Wakayama. Wreath determinants for group-subgroup pairs. *Preprint*, 2014. [arXiv:1406.2425](https://arxiv.org/abs/1406.2425)
6. Shlomo Hoory, Nathan Linial and Avi Wigderson. Expander graphs and their applications. *Bulletin (New Series) of the American Mathematical Society*, 43(4):439–561, October 2006.
7. Kazufumi Kimoto and Masato Wakayama. Invariant theory for singular  $\alpha$ -determinants. *Journal of Combinatorial Theory, Series A*, 115(1):1–31, 2008.
8. Ulrich Knauer. *Algebraic Graph Theory: morphisms, monoids, and matrices*. De Gruyter studies in mathematics ; 41, 2011.
9. A. Lubotzky, R. Phillips and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8:261–277, 1988.
10. G. A. Margulis. Explicit group-theoretical constructions of combinatorial schemes and their applications to the design of expanders and concentrator. *Journal Problems of Information Transmission*, 24,1:51–60, 1988.

11. Toshikazu Sunada. *Fundamental groups and Laplacians*. Proc. Taniguchi Symp. "Geometry and Analysis on Manifolds", 1987, Springer Lect. Note in Math. 1339(1988), 248-277
12. Audrey Terras. *Zeta Functions of Graphs - A Stroll through the Garden*. Cambridge studies in advanced mathematics 128, 2011.

Cid Reyes-Bustos

Graduate School of Mathematics

Kyushu University

744 Motooka, Nishi-ku, Fukuoka, 819-0395 JAPAN

ma213054@math.kyushu-u.ac.jp